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A NOTE ON SELF SIMILAR VECTOR FIELDS IN CYLINDRICALLY SYMMETRIC STATIC SPACE-TIMES

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Abstract. A different approach, which consists of algebraic and direct integration techniques, is developed to study self similar vector fields in cylindrically symmetric static space-times. Here we discuss self similar vector fields of first, second, zeroth and infinite kinds for the above space-times. We have shown that for the special class of the above space-times admit proper homothetic vector fields.

Keywords: algebraic and direct integration techniques, self similar vector fields.

AMS Subject Classification: 83C05, 83C15, 83C20.

1. Introduction

A vector field X is said to be self similar if it satisfies the following two conditions [6]

$$\underset{X}{L}\, u_a = \alpha\, u_a, \tag{1}$$

$$\underset{X}{L}\, h_{ab} = 2\,\delta\, h_{ab}. \tag{2}$$

where L represents Lie derivative with respect to the vector field X, u^a is the four-velocity of the fluid satisfying $u^a u_a = \in$ and $h_{ab} = g_{ab} + \in u_a u_b$ is the projection tensor, $\in = \pm 1$ and α, $\delta \in R$. If $\in = 1$ the vector field u^a is said to be space-like otherwise it is time like. If $\delta \neq 0$, the similarity transformation is characterized by the scale-independent ratio α/δ, which is called the similarity index. If the ratio is unity, X turns out to be a homothetic vector field. In the context of self similarity, homothety is referred to as self similarity of the first kind. If $\alpha = 0$ and $\delta \neq 0$, it is referred to as self similarity of the zeroth kind. If the ratio is not equal to zero or one, it is referred to as self similarity of the second kind. If $\alpha \neq 0$ and $\delta = 0$, it is referred to as self similarity of the infinite kind. If $\delta = \alpha = 0$, X turns out to be a Killing vector fields. If a self similar vector field X is in the direction of the four velocity of the fluid then it is said to be non-tilted parallel self similar vector field. If a self similar vector field X is along the hyper surface, then it will be perpendicular to the fluid flow and is called non-tilted orthogonal self similar vector field. If a self similar vector field X is neither orthogonal nor parallel to the fluid flow it is said to be tilted.

Over the past few years much work has been done in studying self similar solutions in some well known space-times [1-4, 7, 9]. Self similar solutions of the Einstein field equations are widely studied for two very important reasons; first, the governing differential equations have some mathematical complexity which is often reduced by the assumption of self similarity and

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the system of partial differential equations is reduced to ordinary differential equations. Second, self similarity solutions are extensively used for cosmological perturbations, star formation, gravitational collapse, primordial black holes, cosmological voids and cosmic censorship [8].

Throughout in this paper M represents a four dimensional, connected, Hausdorff space-time manifold with Lorentz metric g of signature (-, +, +, +). The curvature tensor associated with g_{ab}, through the Levi-Civita connection, is denoted in component form by R^a_{bcd}. The usual covariant and partial derivatives are denoted by a semicolon and a comma respectively. Here, M is assumed non flat in the sense that the curvature tensor does not vanish over any non empty open subset of M.

2. Main Results

Consider the cylindrically symmetric static space-times in usual coordinate system (t, r, θ, z) (labeled by (x^0, x^1, x^2, x^3), respectively) with the line element

$$ds^2 = -e^{A(r)}dt^2 + dr^2 + e^{B(r)}d\theta^2 + e^{C(r)}dz^2, \tag{3}$$

where A, B and C are arbitrary functions of r only. It is to be remember that here we shall take the four-velocity vector as time like vector field and define by $u_a = e^{\frac{A}{2}}\,\delta^0_a$, where $u^a u_a = -1$, thus the line element (3) becomes

$$ds^2 = -du^2 + dr^2 + e^{B(r)}d\theta^2 + e^{C(r)}dz^2. \tag{4}$$

The Ricci tensor Segre type of the above space-times (4) is {1,111} or one of its degeneracies. The above space-times (4) become 1+3 decomposable and admit nowhere zero time-like covariantly constant vector field u_a such that $u_{a;b} = 0$ and $u_a u^a = -1$. Expanding equation (1) and use the fact u_a is covariantly constant we get

$$X^b{}_{,a}\, u_b = \alpha\, u_a \Rightarrow X^0{}_{,a} = \alpha\ u_a. \tag{5}$$

Equation (5) implies that $X^0 = X^0(u)$ and $X^0 = \alpha\ u + \lambda$, where $\lambda \in R$. The non zero components of the projection tensor h_{ab} are

$$h_{11} = 1,\ h_{22} = e^{B(r)} \text{ and } h_{33} = e^{C(r)}. \tag{6}$$

Expanding equation (2) explicitly and using (6) we get

$$X^1{}_{,1} = \delta, \tag{7}$$

$$e^{B(r)}X^2{}_{,1} + X^1{}_{,2} = 0, \tag{8}$$

$$e^{C(r)}X^3{}_{,1} + X^1{}_{,3} = 0, \tag{9}$$

$$B^\bullet(r)X^1 + 2X^2{}_{,2} = 2\delta, \tag{10}$$

$$e^{(C(r)-B(r))}\, X^3{}_{,2} + X^2{}_{,3} = 0, \tag{11}$$

$$C^\bullet(r)X^1 + 2X^3{}_{,3} = 2\delta, \tag{12}$$

where ‘dot’ represents differentiation with respect to r. Equations (7), (8) and (9) give

$$\begin{aligned} &X^1 = \delta\, r + K^1(\theta, z), \\ &X^2 = -K^1_\theta(\theta, z) \int e^{-B(r)} dr + K^2(\theta, z), \\ &X^3 = -K^1_z(\theta, z) \int e^{-C(r)} dr + K^3(\theta, z), \end{aligned} \tag{13}$$

where $K^1(\theta, z)$, $K^2(\theta, z)$ and $K^3(\theta, z)$ are functions of integration.

Considering equation (11) and using equation (13) we get

$$e^{C-B}\left(-K^1_{z\theta}(\theta, z)\int e^{-C(r)}dr + K^3_\theta(\theta, z)\right) + \left(-K^1_{\theta z}(\theta, z)\int e^{-B(r)}dr + K^2_z(\theta, z)\right) = 0.$$

Differentiating the above equation with respect to r and after some simplification we have

$$\left((C-B)^\bullet e^C \int e^{-C(r)}dr + 2\right) K^1_{z\theta}(\theta, z) - (C-B)^\bullet e^C K^3_\theta(\theta, z) = 0.$$

Now divided above equation with $(C-B)^\bullet e^C$ and assuming that $(C-B)^\bullet \neq 0$ (The case when $(C-B)^\bullet = 0 \Rightarrow B = C + \text{constant}$ the above space-times (4) become plane symmetric and their self similar solutions are given in [5]) and differentiating again with respect to r we get $\left(\int e^{-C(r)}dr + \frac{2e^{-C(r)}}{(C-B)^\bullet}\right)^\bullet \neq 0$ and $K^1_{z\theta}(\theta, z) = 0$. Substituting back we get $K^2_z(\theta, z) = 0$ and $K^3_\theta(\theta, z) = 0$. Equations $K^1_{z\theta}(\theta, z) = 0$, $K^2_z(\theta, z) = 0$ and $K^3_\theta(\theta, z) = 0$ give $K^1(\theta, z) = F^1(\theta) + F^2(z)$, $K^2(\theta, z) = F^4(\theta)$ and $K^3(\theta, z) = F^3(z)$, respectively, where $F^1(\theta)$, $F^2(z)$, $F^4(\theta)$ and $F^3(z)$ are functions of integration. Equation (13) becomes

$$\begin{aligned} &X^1 = \delta\, r + F^1(\theta) + F^2(z), \qquad X^2 = -F^1_\theta(\theta)\int e^{-B(r)}dr + F^4(\theta), \\ &X^3 = -F^2_z(z)\int e^{-C(r)}dr + F^3(z). \end{aligned} \tag{14}$$

Consider equation (10) and differentiating with respect to r and using (14) we get $B^\bullet F^2_z(z) = 0$. There exists the following three possibilities which are:

(a) $B^\bullet \neq 0$, $F^2_z(z) = 0$ (b) $B^\bullet = 0$, $F^2_z(z) \neq 0$ (c) $B^\bullet = 0$, $F^2_z(z) = 0$.

First consider (a), equation $F^2_z(z) = 0 \Rightarrow F^2(z) = c_1$, where $c_1 \in R$. Now consider equation (12) and differentiating with respect to θ and using (14) we have $C^\bullet F^1_\theta(\theta) = 0$. Again there exist three possibilities which are:

(d) $C^\bullet \neq 0$, $F^1_\theta(\theta) = 0$ (e) $C^\bullet = 0$, $F^1_\theta(\theta) \neq 0$ (f) $C^\bullet = 0$, $F^1_\theta(\theta) = 0$.

In **case (ad)** we have $B^\bullet \neq 0$, $C^\bullet \neq 0$, $F^1_\theta(\theta) = 0$ and $F^2(z) = c_1$. Equation $F^1_\theta(\theta) = 0 \Rightarrow$ $\Rightarrow F^1(\theta) = c_2$, where $c_2 \in R$ and $c_1 + c_2 = d_1$. Consider equation (10) and use the above information we get $B^\bullet(\delta\, r + d_1) + 2F^4_\theta(\theta) = 2\delta$. Differentiating with respect to θ we get $F^4_{\theta\theta}(\theta) = 0 \Rightarrow F^4(\theta) = \theta\, d_2 + d_4$, where $d_2, d_4 \in R$. Substituting back we get $B = \ln(\delta\, r + d_1)^{2(1-\frac{d_2}{\delta})}$. Now consider equation (12) and use the above information we get $C^\bullet(\delta\, r + d_1) + 2F^3_z(z) = 2\delta$. Differentiating with respect to z we get $F^3_{zz}(\theta) = 0 \Rightarrow F^3(z) = z\, d_3 + d_5$, where $d_3, d_5 \in R$. Substituting back we get $C = \ln(\delta\, r + d_1)^{2(1-\frac{d_3}{\delta})}$. In this case the solution of the above equations from (7) to (12)

$$X^1 = \delta\, r + d_1, \quad X^2 = \theta\, d_2 + d_4, \quad X^3 = z\, d_3 + d_5. \tag{15}$$

In **case (ae)** we have $B^\bullet \neq 0$, $C^\bullet = 0$, $F^1_\theta(\theta) \neq 0$ and $F^2(z) = c_1$. If one proceeds further like the previous case one find that $B^\bullet = 0$ which gives contradiction to our assumption. Hence this case is not possible.

In **case (af)** we have $B^\bullet \neq 0$, $C^\bullet = 0$, $F^1_\theta(\theta) = 0$ and $F^2(z) = c_1$. Equations $F^1_\theta(\theta) = 0$ and $C^\bullet = 0 \Rightarrow F^1(\theta) = c_2$ and $C = \beta$ where $c_2, \beta \in R$ and $c_1 + c_2 = d_1$. Consider equation (12) and using the above information and upon integration with respect to z we get $F^3(z) = \delta\, z + d_3$, $d_3 \in R$. Now consider equation (10) and use the above information we get $B^\bullet(\delta\, r + d_1) +$

$+ 2F^4_\theta(\theta) = 2\delta$. Differentiating with respect to θ we get $F^4_{\theta\theta}(\theta) = 0 \Rightarrow F^4(\theta) = \theta\, d_2 + d_4$, where $d_2, d_4 \in R$. Substituting back we get $B = \ln(\delta\, r + d_1)^{2(1 - \frac{d_2}{\delta})}$. In this case the solution of the above equations from (7) to (12)

$$X^1 = \delta\, r + d_1, \quad X^2 = \theta\, d_2 + d_4, \quad X^3 = z\,\delta + d_3. \tag{16}$$

In **case (b)** we have $B^\bullet = 0$ and $F^2_z(z) \neq 0$. Equation $B^\bullet = 0 \Rightarrow B = \eta$, where $\eta \in R$. If one proceeds further one find that $F^2_z(z) = 0$ which gives contradiction to our assumption. Hence this case is not possible.

In **case (c)** we have $B^\bullet = 0$ and $F^2_z(z) = 0$. Equations $B^\bullet = 0$ and $F^2_z(z) = 0 \Rightarrow B = \eta$ and $F^2(z) = c_1$, where $c_1, \eta \in R$. Consider equation (10) and use the above information we have $-2\, r\, e^{-\eta}\, F^1_{\theta\theta}(\theta) + 2F^4_\theta(\theta) = 2\delta$. Differentiating with respect to r one has $F^1_{\theta\theta}(\theta) = 0$ and $F^4_\theta(\theta) = \delta \Rightarrow F^1(\theta) = \theta\, c_3 + c_2$ and $F^4(\theta) = \delta\,\theta + d_4$, where $c_2, c_3, d_4 \in R$. Now consider equation (12) and use the above information we get $B^\bullet\,(\delta\, r + \theta\, c_3 + d_1) + 2F^3_z(z) = 2\delta$, where $c_1 + c_2 = d_1$. Differentiating with respect to θ we get $c_3 = 0$. Again differentiating with respect to z one finds that $F^3_{zz}(z) = 0 \Rightarrow F^3(z) = z\, d_3 + d_5$, where $d_3, d_5 \in R$. Substituting back we get $C = \ln(\delta\, r + d_1)^{2(1 - \frac{d_3}{\delta})}$. In this case the solution of the above equations from (7) to (12)

$$X^1 = \delta\, r + d_1, \quad X^2 = \delta\,\theta + d_4, \quad X^3 = z\, d_3 + d_5. \tag{17}$$

It is important to note that from the above calculations it follows that there exists three possibilities when the above equations (7) to (12) admit solution these are

(ad) $B = B(r)$ and $C = C(r)$. (c) $B =$ constant and $C = C(r)$.

(af) $B = B(r)$ and $C =$ constant.

In order to further classification we will discuss each possibility in turn.

Case (ad): Here, we have $B = B(r)$ and $C = C(r)$. In this case the above space-times (4) become

$$ds^2 = -du^2 + dr^2 + (\delta\, r + d_1)^{2(1 - \frac{d_2}{\delta})} d\theta^2 + (\delta\, r + d_1)^{2(1 - \frac{d_3}{\delta})} dz^2. \tag{18}$$

The solution of the above equation from (7) to (12) is given in equation (15). It is important to remind the reader that $X^0 = \alpha\, u + \lambda$. Now for this case we will discuss tilted and non tilted self similar vector fields:

Tilted Cases

There exist following three possibilities which we discuss in turn.

(a) Here we choose $\alpha \neq 0$, $\delta \neq 0$ and $\alpha = \delta$. The line element takes the form

$$ds^2 = -du^2 + dr^2 + (\alpha\, r + d_1)^{2(1 - \frac{d_2}{\alpha})}\, d\theta^2 + (\alpha\, r + d_1)^{2(1 - \frac{d_3}{\alpha})} dz^2. \tag{19}$$

Self similar vector fields take the form

$$X^0 = \alpha\, u + \lambda, \;\; X^1 = \delta\, r + d_1, \quad X^2 = \theta\, d_2 + d_4, \quad X^3 = z\, d_3 + d_5. \tag{20}$$

The vector field in (20) is tilted to the time-like vector field u^a and gives the self similarity of first kind.

(b) In this case we choose $\alpha \neq 0$, $\delta \neq 0$ and $\alpha \neq \delta$. The line element in this case takes the form

$$ds^2 = -du^2 + dr^2 + (\delta\, r + d_1)^{2(1 - \frac{d_2}{\delta})}\, d\theta^2 + (\delta\, r + d_1)^{2(1 - \frac{d_3}{\delta})} dz^2. \tag{21}$$

Self similar vector fields take the form

$$X^0 = \alpha\, u + \lambda, \;\; X^1 = \delta\, r + d_1, \quad X^2 = \theta\, d_2 + d_4, \quad X^3 = z\, d_3 + d_5. \tag{22}$$

The vector field in this case is tilted to the time-like vector u^a and represents the self similarity of second kind.

(c) Now we choose $\alpha \neq 0$ and $\delta = 0$. Here, the line element becomes

$$ds^2 = -du^2 + dr^2 + e^{(a_1 r + c_1)} d\theta^2 + e^{(a_2 r + c_2)} dz^2, \tag{23}$$

where $a_1, a_2, c_1, c_2 \in R$. Self similar vector fields become

$$X^0 = \alpha u + \lambda, \quad X^1 = d_1, \quad X^2 = \theta d_2 + d_4, \quad X^3 = z d_3 + d_5, \tag{24}$$

where $a_1 = -2\frac{d_2}{d_1}$, $a_2 = -2\frac{d_3}{d_1}$ and $d_1, d_2, d_3 \in R (d_1 \neq 0)$. Here the vector field is tilted to the time-like vector u^a and represents the self similarity of Infinite kind.

Non Tilted Case:

If we choose $\alpha = 0$, and $\delta \neq 0$. The line element becomes

$$ds^2 = -du^2 + dr^2 + (\delta r + d_1)^{2(1 - \frac{d_2}{\delta})} d\theta^2 + (\delta r + d_1)^{2(1 - \frac{d_3}{\delta})} dz^2, \tag{25}$$

where $d_1, d_2, d_3 \in R$. Self similar vector fields take the form

$$X^0 = \lambda, \quad X^1 = \delta r + d_1, \quad X^2 = \theta d_2 + d_4, \quad X^3 = z d_3 + d_5. \tag{26}$$

where $d_4, d_5 \in R$. In this case the vector field is non-tilted perpendicular to the time-like vector u^a and represents the self similarity of zeroth kind.

Case (c): In this case we have $B =$ constant and $C = C(r)$. If one proceeds further one finds that the above space-times (4) become

$$ds^2 = -du^2 + dr^2 + d\theta^2 + (\delta r + d_1)^{2(1 - \frac{d_3}{\delta})} dz^2. \tag{27}$$

The solution of the above equation from (7) to (12) is given in equation (17). It is important to remind the reader that $X^0 = \alpha u + \lambda$. Now we will discuss tilted and non tilted self similar vector fields in this case:

Tilted Case:

In this case if we choose $\alpha \neq 0$, $\delta \neq 0$, $\alpha = \delta$ and the line element becomes

$$ds^2 = -du^2 + dr^2 + d\theta^2 + (\alpha r + d_1)^{2(1 - \frac{d_3}{\alpha})} dz^2. \tag{28}$$

Self similar vector fields become

$$X^0 = \alpha u + \lambda, \quad X^1 = \alpha r + d_1, \quad X^2 = \alpha \theta + d_4, \quad X^3 = z d_3 + d_5. \tag{29}$$

The vector field in this case is tilted to the time-like vector u^a and represents the self similarity of First kind. If we choose $\alpha \neq 0$, $\delta \neq 0$ and $\alpha \neq \delta$. The line element becomes

$$ds^2 = -du^2 + dr^2 + d\theta^2 + (\delta r + d_1)^{2(1 - \frac{d_3}{\delta})} dz^2. \tag{30}$$

The Self similarity of second kind becomes

$$X^0 = \alpha u + \lambda, \quad X^1 = \delta r + d_1, \quad X^2 = \delta \theta + d_4, \quad X^3 = z d_3 + d_5. \tag{31}$$

The vector field in this case is tilted to the time-like vector u^a and represents the self similarity of second kind. Now if we choose $\alpha \neq 0$ and $\delta = 0$, we get

$$X^0 = \alpha u + \lambda, \quad X^1 = d_1, \quad X^2 = d_4, \quad X^3 = z d_3 + d_5 \tag{32}$$

with line element

$$ds^2 = -du^2 + dr^2 + d\theta^2 + e^{(a_3 r + c_3)} dz^2, \tag{33}$$

where $a_3 = -2\frac{d_4}{d_1}$, $d_1 \neq 0$ and $a_3, c_3 \in R$. The vector field in this case is tilted to the time-like vector u^a and represents the self similarity of Infinite kind.

Non Tilted Case:

In this case if we choose $\alpha = 0$ and $\delta \neq 0$ we get

$$X^0 = \lambda, \quad X^1 = \delta\, r + d_1, \quad X^2 = \delta\, \theta + d_4, \quad X^3 = z\, d_3 + d_5 \tag{34}$$

with line element

$$ds^2 = -du^2 + dr^2 + d\theta^2 + (\delta\, r + d_1)^{2(1-\frac{d_3}{\delta})} dz^2. \tag{35}$$

In this case the vector field is non-tilted perpendicular to the time-like vector u^a and represents the self similarity of zeroth kind. The case (af) is exactly the same.

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